1. Interior estimate

Theorem 1. *Suppose that a smooth function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies $u_t = u_{xx}$ and $|u(x,t)| \leqslant M$ for all $(x,t) \in \mathbb{R} \times [0,T)$ and some constant $M > 0$. Then, for $t > 0$ we have*

$$u_x^2 \leqslant \tfrac{t+1}{2t} M^2. \tag{1}$$

Proof. Given small $\epsilon > 0$, we define a cut-off function

$$\eta(x,t) = \max\{0, 1 - \tfrac{1}{6}\epsilon(x^2 + 2t)\}, \tag{2}$$

and define a test function

$$w(x,t) = \tfrac{t}{t+1}\eta^2 u_x^2 + (\tfrac{1}{2} + \epsilon)u^2 - (\tfrac{1}{2} + 2\epsilon)M^2 - \epsilon t, \tag{3}$$

which satisfies $w(x,0) < 0$.

We observe that w is smooth in the support $\operatorname{supp}(\eta) = \{(x,t) : \eta(x,t) > 0\}$. We compute

$$w_t = (t+1)^{-2}\eta^2 u_x^2 + \tfrac{2t}{t+1}(\eta\eta_t u_x^2 + u_x u_{xt}\eta^2) + (1 + 4\epsilon)uu_t - \epsilon. \tag{4}$$

and

$$w_{xx} = \tfrac{2t}{t+1}(\eta\eta_{xx}u_x^2 + \eta_x^2 u_x^2 + 2\eta\eta_x u_x u_{xx} + u_x u_{xxx}\eta^2 + u_{xx}^2\eta^2) + (1 + 4\epsilon)(uu_{xx} + u_x^2). \tag{5}$$

Thus, $(1+t)^{-2}\eta^2 \leqslant 1$ and $\eta_t = \eta_{xx}$ yields

$$w_t - w_{xx} \leqslant -\epsilon - 4\epsilon u_x^2 - \tfrac{2t}{t+1}(\eta_x^2 u_x^2 + 2\eta\eta_x u_x u_{xx} + u_{xx}^2\eta^2) \tag{6}$$

$$< -4\epsilon u_x^2 - \tfrac{6t}{t+1}\eta_x^2 u_x^2 \leqslant -4\epsilon u_x^2 - 4\epsilon(\tfrac{1}{6}\epsilon x^2)u_x^2. \tag{7}$$

Since $\frac{1}{6}\epsilon x^2 \leqslant 1$ holds in $\operatorname{supp}(\eta)$, $w_t < w_{xx}$ holds in $\operatorname{supp}(\eta)$.

Now, we claim $w \leqslant 0$. If not, there exists a space-time point $(x_0, t_0) \in \operatorname{supp}(\eta)$ with $t_0 > 0$ such that $w(x,t) < 0$ holds for all $t \in [0, t_0)$ and $w(x_0, t_0) = 0$. Then, we have a contradiction as follows.

$$0 \leqslant w_t(x_0, t_0) < w_{xx}(x_0, t_0) \leqslant 0. \tag{8}$$

Therefore, we have $w \leqslant 0$ in $\operatorname{supp}(\eta)$, namely

$$\tfrac{t}{t+1}[1 - \tfrac{1}{6}\epsilon(x^2 + 2t)]^2 u_x^2 + (\tfrac{1}{2} + \epsilon)u^2 - (\tfrac{1}{2} + 2\epsilon)M^2 - \epsilon t \leqslant 0 \tag{9}$$

holds if $x^2 + 2t \leqslant (6\epsilon)^{-1}$. By passing $\epsilon \to 0$, we have

$$\tfrac{t}{t+1} u_x^2 \leqslant \tfrac{1}{2}(M^2 - u^2) \leqslant \tfrac{1}{2} M^2 \tag{10}$$

for all $(x, t) \in \mathbb{R} \times [0, T)$.

□

2. Barrier

Theorem 2. *Let Ω be a bounded convex open set of $\mathbb{R}^2$. Suppose that a smooth function u satisfies $u_t = \Delta u$ in $\overline{Q}_T$ where $Q_T = \Omega \times [0, T)$, $u(x, 0) = g(x)$ for $x \in \overline{\Omega}$ where $g : \overline{\Omega} \to \mathbb{R}$ is smooth, and $u(\sigma, t) = 0$ for $\sigma \in \partial\Omega$ and $t \in [0, T)$. Then, the following holds in $\overline{Q}_T$.*

$$\|\nabla u(\vec{x}, t)\| \leqslant \max_{\vec{x} \in \overline{\Omega}} \|\nabla g(\vec{x})\|. \tag{11}$$

Proof. Suppose that $\vec{0} \in \partial\Omega$ and $\nu(0) = -e_1$. We define an upper barrier $w = Kx_1$, where $K = \max \|\nabla g\|$, and consider $v = u - w$. Since $v \leqslant 0$ holds on $\partial_p Q_T$ and $v_t = v_{xx}$ in $\overline{Q}_T$, the maximum principle implies $v \leqslant 0$ in $\overline{Q}_T$. Hence

$$u_1(\vec{0}, t) = \lim_{x_1 \to 0} \frac{u(x_1, 0, t) - u(0, 0, t)}{x_1} \leqslant \lim_{x_1 \to 0} \frac{K x_1}{x_1} = K. \tag{12}$$

Similarly, we have $u_1 \geqslant K$ by using the lower barrier $-Kx_1$, and thus

$$\|\nabla u(\vec{0}, t)\| = \sqrt{|u_1(\vec{0}, t)|^2 + |u_2(\vec{0}, t)|^2} \leqslant |u_1(\vec{0}, t)| \leqslant K. \tag{13}$$

By rotating and shifting the coordinate system, we have $\|\nabla u(\sigma, t)\| \leqslant K$ for each $\sigma \in \partial\Omega$. Since $\|\nabla u(\vec{x}, 0)\| = \|\nabla g(\vec{x})\| \leqslant K$,

$$\|\nabla u\| \leqslant K = \max \|g(x, 0)\|, \tag{14}$$

holds on $\partial_p Q_T$.

Next, we define $v = \|\nabla u\|^2 - K^2$, which satisfies $v \leqslant 0$ on $\partial_p Q_T$. Moreover,

$$\Delta v = 2\|\nabla^2 u\|^2 + 2\langle \nabla u, \nabla \Delta u \rangle \geqslant 2\langle \nabla u, \nabla u_t \rangle = v_t, \tag{15}$$

where $\|\nabla^2 u\|$ denotes the square root norm of the matrix $\nabla^2 u$, namely

$$\|\nabla^2 u\|^2 = \sum_{i,j} u_{ij}^2. \tag{16}$$

Therefore, the maximum principle implies the desired result $v \leqslant K^2$ in $\overline{Q}_T$. □

3. Backward uniqueness

Theorem 3 (Hölder inequality)**.** *Suppose the f^2 and g^2 are integrable functions on $\Omega \subset \mathbb{R}^n$. Then, the following holds*

$$\int_\Omega f(\vec{x})g(\vec{x})d\vec{x} \leqslant \sqrt{\int_\Omega |f(\vec{x})|^2 d\vec{x}} \sqrt{\int_\Omega |g(\vec{x})|^2 d\vec{x}}. \tag{17}$$

Theorem 4. *Let $Q_T = [0, L] \times [0, T]$. u are v are solutions to the heat equation over Q_T. Suppose $u = v$ holds on $[0, L] \times \{T\}$ and $\{0, L\} \times [0, T]$. Then, $u = v$.*

Proof. We define $w = u - v$. Then, $w = 0$ on $\{0, L\} \times [0, T]$. We recall the energy

$$E(t) = \int_0^L w^2(x, t)dx. \tag{18}$$

Then,

$$E' = -2\int_0^L ww_{xx}dx, \qquad E'' = 4\int_0^L w_{xx}^2 dx. \tag{19}$$

Hence, the Hölder inequality implies

$$E'^2 \geqslant EE''. \tag{20}$$

On the other hand, $E' \leqslant 0$ and $E(T) = 0$ implies that there exist some $\tilde{T} \in [0, T]$ such that $E(t) > 0$ for $t < \tilde{T}$ and $E(t) = 0$ for $t \geqslant \tilde{T}$. If $\tilde{T} = 0$ then $w = 0$. Hence, we may assume $\tilde{T} > 0$ towards a contradiction. Then, for $t \in [0, \tilde{T})$ we have

$$(\log E(t))'' = E^{-2}(E''E - E'^2) \geqslant 0. \tag{21}$$

Thus, we have

$$(\log E(t))' \geqslant (\log E(0))', \tag{22}$$

and thus

$$\log E(t) \geqslant \log E(0) + (\log E(0))'t. \tag{23}$$

Passing t to $\tilde{T}$ yields a contradiction.

$$-\infty = \lim_{t \to \tilde{T}} \log E(t) \geqslant \log E(0) + (\log E(0))'\tilde{T} \geqslant -C. \tag{24}$$

□

4. Li-Yau type Harnack inequality I

Theorem 5. *$u : \mathbb{R}\times[-T,0]$ is a positive smooth solution to the heat equation, and $u(x+L,t) = u(x,t)$ holds for some $L > 0$. Then, the following holds*

$$u_t \geqslant -\tfrac{1}{2}(t+T)^{-1}. \tag{25}$$

In particular, if $-T = -\infty$ then $u_t \geqslant 0$.

Proof. Since u is positive, $v = \log u$ is a well-defined smooth function. In addition, v satisfies

$$v_t = v_{xx} + v_x^2. \tag{26}$$

Differentiating the equation twice yields

$$v_{txx} = v_{xxxx} + 2v_{xxx}v_x + 2v_{xx}^2. \tag{27}$$

Given $\epsilon > 0$, we define $w = v_{xx} + (\frac{1}{2}+\epsilon)(t+T)^{-1}$ for $t > -T$. Then, the equation above implies

$$w_t = w_{xx} + 2w_x v_x + 2v_{xx}^2 - (\tfrac{1}{2}+\epsilon)(t+T)^{-2}. \tag{28}$$

We claim that $w \geqslant 0$ for all $t > -T$. Since w is periodic and $\lim_{t\to -T} w(x,t) = +\infty$ holds for all $x \in \mathbb{R}$, if $w < 0$ at some point then there exists some space-time point $(x_0,t_0) \in \mathbb{R}\times(-T,0]$ such that $w(x_0,t_0) = 0$ and $w(x,t) > 0$ for all $(x,t) \in \mathbb{R}\times(-T,t_0)$. Therefore, at the minimum point (x_0,t_0), we have

$$0 \geqslant 2v_{xx}^2(x_0,t_0) - (\tfrac{1}{2}+\epsilon)(t_0+T)^{-2}. \tag{29}$$

In addition, we have

$$0 = w(x_0,t_0) = v_{xx}(x_0,t_0) + (\tfrac{1}{2}+\epsilon)(t_0+T)^{-1}. \tag{30}$$

Combing them yields a contradiction.

$$0 \geqslant 2(\tfrac{1}{2}+\epsilon)^2(t_0+T)^{-2} - (\tfrac{1}{2}+\epsilon)(t_0+T)^{-2} = 2\epsilon(\tfrac{1}{2}+\epsilon)(t_0+T)^{-2} > 0. \tag{31}$$

Namely,

$$v_{xx} \geqslant -(\tfrac{1}{2}+\epsilon)(t+T)^{-1}, \tag{32}$$

holds for all $\epsilon > 0$. Hence, passing $\epsilon \to 0$ and $\frac{u_t}{u} = v_t = v_{xx} + v_x^2 \geqslant v_{xx}$ yield the desired result. □